

EXACT SOLUTIONS OF GENERALIZED
CALOGERO-SUTHERLAND MODELS
— BC_N and C_N cases —Masamichi Kojima¹ and Nobuyoshi Ohta²*Department of Physics, Osaka University,
Toyonaka, Osaka 560, Japan***Abstract**

Using a collective field method, we obtain explicit solutions of the generalized Calogero-Sutherland models that are characterized by the roots of the classical groups B_N and C_N . Starting from the explicit wave functions for A_{N-1} type expressed in terms of the singular vectors of the W_N algebra, we give a systematic method to construct wave functions and derive energy eigenvalues for other types of theories.

¹e-mail address: koji2@phys.wani.osaka-u.ac.jp²e-mail address: ohta@phys.wani.osaka-u.ac.jp

The Calogero-Sutherland (CS) models [1] describe one-dimensional quantum systems of N_0 particles on a circle interacting with each other by an inverse square potential. The complete excitation spectrum and wave functions are exactly calculable in these models. They play significant roles in various subjects such as fractional statistics [2, 3], quantum Hall effect [4] and W_∞ algebra [5].

Although it had been difficult to solve this eigenvalue problem directly, Stanley and Macdonald [6] found that the solutions are expressed by Jack symmetric polynomials and studied their properties. However, they did not show how to construct Jack polynomials, which are necessary to calculate correlation functions explicitly. Thus it is an important problem to find a systematic method to construct Jack polynomials.

Recently this problem has been solved by the use of collective field method [7] and conformal field theory technique by Awata et al. [8]. They have shown that the Hamiltonian can be expressed in terms of Virasoro and W_N generators of positive modes and hence the Jack symmetric polynomials can be represented as W_N singular vectors, whose explicit forms are given by integral representations using free bosons.

Among many variants of the CS models [2], a class of models have been known to be exactly solvable and show interesting behaviors similar to the original ones. They are the Lie-algebraic generalization of the above models [9]. In particular the so-called CS model of BC_N -type (hereafter referred to as BC_N -CS model) is the most general one with N_0 interacting particles. This model is known to be relevant to one-dimensional physics with boundaries. The energy eigenvalues for these models have been obtained for both ground and excited states [10, 11], but the wave functions have been known only for the ground states [9, 11]. The purpose of this paper is to give a systematic method to construct the wave functions for excited states explicitly and also give elementary derivation of the energy eigenvalues by using the collective field method.

The Hamiltonian H_{CS} for ordinary CS models is given by

$$H_{CS} = -\sum_{i=1}^{N_0} \frac{1}{2} \frac{\partial^2}{\partial q_i^2} + \left(\frac{\pi}{L}\right)^2 \sum_{\substack{i,j=1 \\ i < j}}^{N_0} \frac{\beta(\beta-1)}{\sin^2 \frac{\pi}{L}(q_i - q_j)}, \quad (1)$$

where L and β are the circumference of the circle and a coupling constant, respectively.

The ground state Ψ_0 and the energy eigenvalue E_0 of H_{CS} is given by [1, 9]

$$\begin{aligned}\Psi_0 &\equiv \Delta_{CS}^\beta = \left(\frac{L}{\pi} \prod_{\substack{i,j=1 \\ i < j}}^{N_0} \sin \frac{\pi}{L} (q_i - q_j) \right)^\beta, \\ E_0 &= \frac{\beta^2}{6} \left(\frac{\pi}{L} \right)^2 (N_0^3 - N_0).\end{aligned}\tag{2}$$

Note that the ground state exhibits fractional statistics for rational β . The excited states of H_{CS} take the form $\Psi = \Delta_{CS}^\beta J_\lambda(q; \beta)$. The functions $J_\lambda(q; \beta)$ are known as the Jack polynomials characterized by an index λ of the Young diagram and are symmetric in the coordinates q_i so that the statistics of the system is determined by the wave function of the ground state.

Looking at the structure of the above Hamiltonian (1), one immediately recognizes that there is a close relation of this model to the root system of the classical group A_{N-1} . It is then natural to consider Lie-algebraic generalization of this model. Indeed, it has been known for some time [9] that the models described by the following Hamiltonian are exactly solvable:

$$H_{GCS} = - \sum_{i=1}^{N_0} \frac{1}{2} \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \left(\frac{\pi}{L} \right)^2 \sum_{\vec{\alpha} \in R_+} \frac{\mu_\alpha (\mu_\alpha + 2\mu_{2\alpha} - 1) |\vec{\alpha}|^2}{\sin^2 \frac{\pi}{L} (\vec{\alpha} \cdot \vec{q})},\tag{3}$$

where R_+ stands for positive roots of the classical group under consideration and the coupling constants μ_α are equal for the roots of the same length. The most general model then is the one with all the roots in B_N and C_N algebras. This is the BC_N -CS model we are going to discuss.

To render our subsequent calculations simple, we introduce the following variables:

$$x_j \equiv \exp \left(\frac{2\pi i q_j}{L} \right); \quad D_i \equiv x_i \frac{\partial}{\partial x_i}.\tag{4}$$

Using these variables, the Hamiltonian (3) is cast into

$$\begin{aligned}H_{GCS} &= \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \left[\sum_{i=1}^{N_0} D_i^2 - 2\beta(\beta - 1) \sum_{\substack{i,j=1 \\ i < j}}^{N_0} \left(\frac{x_i x_j}{(x_i - x_j)^2} + \frac{x_i x_j^{-1}}{(x_i - x_j^{-1})^2} \right) \right. \\ &\quad \left. - \sum_{i=1}^{N_0} \left(\gamma(\gamma + 2\delta - 1) \frac{x_i}{(x_i - 1)^2} + 4\delta(\delta - 1) \frac{x_i^2}{(x_i^2 - 1)^2} \right) \right],\end{aligned}\tag{5}$$

where we have used β, γ, δ for coupling constants. We note that putting $\gamma = 0$ reduces the model to C_N -type, $\delta = 0$ to B_N -type, and finally $\gamma = \delta = 0$ to D_N -type. In this paper, we will be mainly concerned with solutions common to all these root systems. The solutions for B_N and D_N contain additional special ones corresponding to spinor representations, which will be discussed in a separate paper.

In analogy to the solution (2), the ground state wave function and energy are given by [9, 11]

$$\begin{aligned}
\Delta_{GCS} &= \prod_{i=1}^{N_0} \left(\sin \frac{\pi}{L} q_i \right)^\gamma \left(\sin \frac{2\pi}{L} q_i \right)^\delta \prod_{\substack{i,j=1 \\ i < j}}^{N_0} \left(\sin \frac{\pi}{L} (q_i - q_j) \sin \frac{\pi}{L} (q_i + q_j) \right)^\beta \\
&= \prod_{i=1}^{N_0} x_i^{-\beta(N_0-1)-\gamma/2-\delta} (x_i - 1)^\gamma (x_i^2 - 1)^\delta \prod_{\substack{i,j=1 \\ i < j}}^{N_0} (x_i - x_j)^\beta (x_i x_j - 1)^\beta \\
E_0^{GCS} &= \sum_{i=1}^{N_0} \left[\frac{\gamma}{2} + \delta + \beta(N_0 - i) \right]^2.
\end{aligned} \tag{6}$$

Now our task is to solve the eigenvalue problem

$$H_{GCS} \Delta_{GCS} \Phi^{GCS} = E_{GCS} \Delta_{GCS} \Phi^{GCS}. \tag{7}$$

The effective Hamiltonian H_{eff} acting on the function $\Phi^{GCS}(x)$ is derived by the transformation

$$\Delta_{GCS}^{-1} H_{GCS} \Delta_{GCS} = \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 [H_{eff} + E_0^{GCS}], \tag{8}$$

and our problem reduces to

$$H_{eff} \Phi^{GCS} = E_{eff} \Phi^{GCS} \quad ; \quad E_{GCS} = \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 [E_0^{GCS} + E_{eff}]. \tag{9}$$

In terms of the variables (4), we find

$$\begin{aligned}
H_{eff} &= \sum_{i=1}^{N_0} D_i^2 + \beta \sum_{\substack{i,j=1 \\ i < j}}^{N_0} \left(\frac{x_i + x_j}{x_i - x_j} (D_i - D_j) + \frac{x_i + x_j^{-1}}{x_i - x_j^{-1}} (D_i + D_j) \right) \\
&\quad + \sum_{i=1}^{N_0} \left(\gamma \frac{x_i + 1}{x_i - 1} + 2\delta \frac{x_i + x_i^{-1}}{x_i - x_i^{-1}} \right) D_i.
\end{aligned} \tag{10}$$

We are going to express this Hamiltonian by free bosons. In particular, it will be related to the free boson representation of the W_N algebra corresponding to the A_{N-1} group.¹ Before

¹Here N is an arbitrary integer (≥ 2) independent of N_0 .

jumping into this, let us first review relevant results in the free boson representation of this algebra [8].

Let \vec{e}_i ($i = 1, \dots, N$) be an orthonormal basis ($\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$). We define the weights of the vector representation \vec{h}_i , the simple roots $\vec{\alpha}^a$ ($a = 1, \dots, N-1$) and the fundamental weights $\vec{\Lambda}_a$ by

$$\begin{aligned} \vec{h}_i &= \vec{e}_i - \frac{1}{N} \sum_{j=1}^N \vec{e}_j, & \vec{\alpha}^a &= \vec{h}_a - \vec{h}_{a+1}, & \vec{\Lambda}_a &= \sum_{i=1}^a \vec{h}_i, \\ \vec{\alpha}^a \cdot \vec{\alpha}^b &\equiv A^{ab} = 2\delta^{a,b} - \delta^{a,b+1} - \delta^{a,b-1}, & \vec{\alpha}^a \cdot \vec{\Lambda}_b &\equiv A_b^a = \delta_b^a. \end{aligned} \quad (11)$$

We then introduce $N-1$ free bosons

$$\vec{\phi}(z) = \sum_{a=1}^{N-1} \phi^a(z) \vec{\Lambda}_a = \sum_{a=1}^{N-1} \phi_a(z) \vec{\alpha}^a. \quad (12)$$

They have the mode expansion

$$\vec{\phi}(z) = \vec{q} + \vec{a}_0 \ln z - \sum_{n \neq 0} \frac{1}{n} \vec{a}_n z^{-n}, \quad (13)$$

with the commutation relations

$$[a_n^a, a_m^b] = A^{ab} n \delta_{n+m,0}, \quad [a_0^a, q^b] = A^{ab}, \quad (14)$$

The boson Fock space is generated by the oscillators of negative modes on the state

$$|\vec{\lambda}\rangle = e^{\vec{\lambda} \cdot \vec{q}} |\vec{0}\rangle; \quad \vec{a}_n |\vec{0}\rangle = 0 \quad (n \geq 0). \quad (15)$$

$\langle \vec{\lambda} |$ is similarly defined with the inner product $\langle \vec{\lambda} | \vec{\lambda}' \rangle = \delta_{\vec{\lambda}, \vec{\lambda}'}$.

The spin 2 and 3 generators of the W_N algebra are given by [12]

$$\begin{aligned} T(z) &\equiv \sum_n L_n z^{-n-2} \\ &= \frac{1}{2} (\partial \vec{\phi}(z))^2 + \alpha_0 \vec{\rho} \cdot \partial^2 \vec{\phi}, \\ W(z) &\equiv \sum_n W_n z^{-n-3} \\ &= \sum_{a=1}^{N-1} (\partial \phi_a(z))^2 (\partial \phi_{a+1}(z) - \partial \phi_{a-1}(z)) \\ &\quad + \alpha_0 \sum_{a,b=1}^{N-1} (1-a) A^{ab} \partial \phi_a(z) \partial^2 \phi_b(z) + \alpha_0^2 \sum_{a=1}^{N-1} (1-a) \partial^3 \phi_a(z), \end{aligned} \quad (16)$$

where

$$\begin{aligned}\alpha_0 &= \sqrt{\beta} - \frac{1}{\sqrt{\beta}}, \\ \vec{\rho} &= \sum_{a=1}^{N-1} \vec{\Lambda}_a; \quad (\vec{\rho})^2 = \frac{1}{12}N(N^2 - 1).\end{aligned}\tag{17}$$

The highest weight states of the W_N algebra are created from the vacuum by the vertex operator as $|\vec{\lambda}\rangle =: e^{\vec{\lambda} \cdot \vec{\phi}(0)} : |\vec{0}\rangle$, whose conformal weight $h(\vec{\lambda})$ and W_0 -eigenvalue $w(\vec{\lambda})$ are

$$\begin{aligned}h(\vec{\lambda}) &= \frac{1}{2} [(\vec{\lambda} - \alpha_0 \vec{\rho})^2 - \alpha_0^2 (\vec{\rho})^2], \\ w(\vec{\lambda}) &= \sum_{a=1}^{N-1} [\lambda_a^2 (\lambda_{a+1} - \lambda_{a-1}) + \alpha_0 (2(a-1)\lambda_a + (1-2a)\lambda_{a+1}) \lambda_a \\ &\quad + 2\alpha_0^2 (1-a)\lambda_a].\end{aligned}\tag{18}$$

If we define $\vec{\lambda}_{\vec{r}, \vec{s}}^{\pm}$ by

$$\begin{aligned}\vec{\lambda}_{\vec{r}, \vec{s}}^+ &= \sum_{a=1}^{N-1} \left[(1 + r^a - r^{a-1})\sqrt{\beta} - (1 + s^a)/\sqrt{\beta} \right] \vec{\Lambda}_a, \\ \vec{\lambda}_{\vec{r}, \vec{s}}^- &= \sum_{a=1}^{N-1} \left[(1 + r^a)\sqrt{\beta} - (1 + s^a - s^{a-1})/\sqrt{\beta} \right] \vec{\Lambda}_a,\end{aligned}\tag{19}$$

singular vectors at level $\sum_{a=1}^{N-1} r^a s^a$ with the highest weight $|\vec{\lambda}_{\vec{r}, \vec{s}}^{\pm}\rangle$ are given as

$$\begin{aligned}|\chi_{\vec{r}, \vec{s}}^+\rangle &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r^a} \frac{dz_j^a}{2\pi i} : e^{\sqrt{\beta} \phi^a(z_j^a)} : |\vec{\lambda}_{\vec{r}, \vec{s}}^+ - \sqrt{\beta} \sum_{a=1}^{N-1} r^a \vec{\alpha}^a\rangle \\ &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r^a} \frac{dz_j^a}{2\pi i z_j^a} \prod_{a=1}^{N-1} \prod_{\substack{i,j=1 \\ i < j}}^{r^a} (z_i^a - z_j^a)^{2\beta} \prod_{a=1}^{N-1} \prod_{i=1}^{r^a} \prod_{j=1}^{r^{a+1}} (z_i^a - z_j^{a+1})^{-\beta} \\ &\quad \times \prod_{a=1}^{N-1} \prod_{i=1}^{r^a} (z_i^a)^{(1-r^a+r^{a+1})\beta-s^a} \prod_{a=1}^{N-1} \prod_{i=1}^{r^a} e^{\sqrt{\beta} \phi_-^a(z_i^a)} |\vec{\lambda}_{\vec{r}, \vec{s}}^+\rangle,\end{aligned}\tag{20}$$

where ϕ_-^a is defined by $\phi_-^a(z) = \sum_{n>0} \frac{1}{n} a_{-n}^a z^n$. One can define similar singular vectors $|\chi_{\vec{r}, \vec{s}}^-\rangle$ at the same level using $\vec{\lambda}_{\vec{r}, \vec{s}}^-$. These singular vectors are annihilated by Virasoro L_n and W_n generators of positive modes and correspond to the following Young diagrams parameterized by the numbers of boxes in each row, $\lambda = (\lambda_1, \dots, \lambda_N)$, $\lambda_1 \geq \dots \geq \lambda_N \geq 0$:

$$\lambda = \begin{array}{|c|c|} \hline s^1 & s^2 \\ \hline r^1 & r^2 \\ \hline \end{array} \dots \dots \begin{array}{|c|c|} \hline s^{N-2} & s^{N-1} \\ \hline r^{N-2} & r^{N-1} \\ \hline \end{array}.$$

We can read off the relation between λ and \vec{r}, \vec{s} from this diagram. We also note that

$$h(\vec{\lambda}_{\vec{r}, \vec{s}}^+ - \sqrt{\beta} \sum_{a=1}^{N-1} r^a \vec{\alpha}^a) = h(\vec{\lambda}_{\vec{r}, \vec{s}}^+) + \sum_{a=1}^{N-1} r^a s^a. \quad (21)$$

Let us give a simple example at level 3 for $\vec{r} = (2, 1, 0, \dots, 0)$; $\vec{s} = (1, 1, 0, \dots, 0)$ which corresponds to the Young diagram $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$. From (20), we find the singular vector is given by

$$|\chi\rangle \propto \left[a_{-3}^1 + \frac{\beta-1}{\sqrt{\beta}} a_{-1}^1 a_{-2}^1 - (a_{-1}^1)^3 - \frac{\beta+2}{2\beta} \left((a_{-1}^1)^2 - \sqrt{\beta} a_{-2}^1 \right) a_{-1}^2 \right] |\lambda\rangle. \quad (22)$$

Coming back to our problem, since our system (10) has the reflection invariance under $x_i \rightarrow x_i^{-1}$ for each i in addition to the permutation symmetry $x_i \leftrightarrow x_j$, we expect that the solutions are invariant under these transformations. It is then natural to define the symmetric power sums^{2,3}

$$p_n = \sum_{i=1}^{N_0} (x_i^n + x_i^{-n}), \quad (23)$$

in terms of which we can express our effective Hamiltonian. We then consider the map

$$\begin{aligned} |f\rangle \mapsto f(x) &\equiv \langle \vec{\lambda} | C_{\beta'} | f \rangle \\ C_{\beta'} &\equiv \exp \left(\beta' \sum_{n>0} \frac{1}{n} a_{n,1} p_n \right), \end{aligned} \quad (24)$$

where β' is a parameter to be determined shortly. This gives the following correspondence between the oscillators and the power sums:

$$\beta' p_n \leftrightarrow a_{-n}^1; \quad \frac{n}{\beta'} \frac{\partial}{\partial p_n} \leftrightarrow a_{n,1}. \quad (25)$$

Note that a_{-n}^a ($a > 1, n > 0$) vanishes under this map.

² In the A_{N-1} case, we define $p_n = \sum_{i=1}^{N_0} x_i^n$. Eq. (23) is the generalization necessary in our system where we do not have translational invariance.

³ It is known in mathematical literature that the representation ring for BC_N and C_N systems is isomorphic to the ring generated by these symmetric power sums. For the special cases of spinor representations in B_N and D_N , we need additional functions $\prod_{i=1}^{N_0} (\sqrt{x_i} + 1/\sqrt{x_i})$ for B_N and $\prod_{i=1}^{N_0} (\sqrt{x_i} - 1/\sqrt{x_i})$ as well for D_N . These special cases will be discussed in a separate paper.

With the help of the map (24), the Hamiltonian (10) is transformed into⁴

$$\begin{aligned}
\hat{H}_{eff} = & \sum_{n,m>0} \left(\beta' a_{-n-m}^1 a_{n,1} a_{m,1} + \frac{\beta}{\beta'} a_{-n}^1 a_{-m}^1 a_{n+m,1} \right) \\
& + \sum_{n>0} [n(1-\beta) + \{\beta(2N_0-1) + \gamma + 2\delta\}] a_{-n}^1 a_{n,1} \\
& - 2\beta' \sum_{n,m>0} a_{-m}^1 a_{n,1} a_{n+m,1} - 2 \sum_{n,m>0} \left\{ (\beta-2\delta) a_{-m}^1 a_{2n+m,1} - \gamma a_{-m}^1 a_{n+m,1} \right\} \\
& - 2\beta' N_0 \sum_{n>0} \left\{ (\beta-2\delta) a_{2n,1} - \gamma a_{n,1} + \beta' (a_{n,1})^2 \right\}. \tag{26}
\end{aligned}$$

Here and in what follows, carets on the Hamiltonian and states mean that they are expressed in terms of oscillators. After straightforward calculation, one finds that this Hamiltonian can, for the choice

$$\beta' = \sqrt{\beta}, \tag{27}$$

be finally rewritten as

$$\begin{aligned}
\hat{H}_{eff} = & \hat{H}' + \sum_{n>0} \hat{H}_n + \sum_{a>1} \sum_{n>0} a_{-n}^a (\cdots) \\
& + \sqrt{\beta} \sum_{n>0} \left(\frac{2}{N_0} a_{-n}^1 L_n - 2a_{n,1} L_n \right) + 2 \sum_{n>0} \{ \gamma L_n + (2\delta - \beta) L_{2n} \}, \tag{28}
\end{aligned}$$

where

$$\begin{aligned}
\hat{H}' = & \sum_{n>0} \vec{a}_{-n} \cdot \vec{a}_n \left(2N_0\beta - 1 + \gamma + 2\delta - 2\sqrt{\beta} a_{0,1} \right) + \sqrt{\beta} (W_0 - W_{0,zero}), \\
\hat{H}_n = & 2\gamma \sum_{a=1}^{N-1} \sum_{m=1}^{n-1} a_{n-m,a} (a_{m,a+1} - a_{m,a}) + \frac{2\gamma}{\sqrt{\beta}} a_{n,1} \left\{ (n+1)(\beta-1) + N_0\beta - \sqrt{\beta} a_0^1 \right\} \\
& + \frac{2\gamma}{\sqrt{\beta}} \sum_{a=2}^{N-1} a_{n,a} \left\{ (n+1)(\beta-1) - \sqrt{\beta} a_0^a \right\} \\
& + 2\sqrt{\beta} \sum_{a=1}^{N-1} \sum_{m=1}^{n-1} a_{n,1} (a_{n-m,a} a_{m,a} - a_{n-m,a+1} a_{m,a}) \\
& + 2(\beta-2\delta) \sum_{a=1}^{N-1} \sum_{m=1}^{2n-1} a_{2n-m,a} (a_{m,a} - a_{m,a+1}) \\
& - 2(a_{n,1})^2 \left\{ (n+1)(\beta-1) + N_0\beta - \sqrt{\beta} a_0^1 \right\}
\end{aligned}$$

⁴It appears that this Hamiltonian (26) is not Hermitian. This is simply because we have transformed our Hamiltonian by the ground state (see eq. (8)).

$$\begin{aligned}
& -2 \sum_{a=2}^{N-1} a_{n,1} a_{n,a} \left\{ (n+1)(\beta-1) - \sqrt{\beta} a_0^a \right\} \\
& - \frac{2}{\sqrt{\beta}} \left[(\beta-2\delta) a_{2n,1} \left\{ (2n+1)(\beta-1) + N_0\beta - \sqrt{\beta} a_0^1 \right\} + \beta a_{2n,1} \right] \\
& - \frac{2}{\sqrt{\beta}} (\beta-2\delta) \sum_{a=2}^{N-1} a_{2n,a} \left\{ (2n+1)(\beta-1) - \sqrt{\beta} a_0^a \right\}.
\end{aligned} \tag{29}$$

Here $W_{0,zero}$ in \hat{H}' is the zero mode part of W_0 . The third term involving a_{-n}^a ($a > 1, n > 0$) in (28) vanishes after multiplying by $\langle \vec{\lambda} | C_{\beta'}$ and will be disregarded in the following. Note that \hat{H}' is the sum of number operators and W_N zero mode and also that \hat{H}_n consist of annihilation operators only.

To construct our eigenstates of the Hamiltonian \hat{H}_{eff} , consider singular vectors at the level $\sum_{a=1}^{N-1} r^a s^a$. Since these are annihilated by Virasoro generators L_n of positive modes, only the first two terms in (28) are relevant to our problem. These are already eigenstates of \hat{H}' with the eigenvalue

$$\begin{aligned}
E_\lambda &= \left[h \left(\vec{\lambda}_{\vec{r},\vec{s}}^+ - \sqrt{\beta} \sum_{a=1}^{N-1} r^a \vec{\alpha}^a \right) - h \left(\vec{\lambda}_{\vec{r},\vec{s}}^+ \right) \right] \left[2N_0\beta - 1 + \gamma + 2\delta - 2 \left(\beta r_1 - s_1 + \sqrt{\beta} \alpha_0 \rho_1 \right) \right] \\
&+ \sqrt{\beta} \left[w \left(\vec{\lambda}_{\vec{r},\vec{s}}^+ - \sqrt{\beta} \sum_{a=1}^{N-1} r^a \vec{\alpha}^a \right) - w \left(\vec{\lambda}_{\vec{r},\vec{s}}^+ \right) \right] \\
&= \sum_{a=1}^{N-1} r^a s^a s^a + 2 \sum_{\substack{a,b=1 \\ a>b}}^{N-1} r^a s^a s^b + \sum_{a=1}^{N-1} r^a s^a (2N_0\beta - \beta + \gamma + 2\delta - \beta r^a), \\
&= \sum_{i=1}^{N_0} \left[\lambda_i^2 + 2 \left\{ \beta(N_0 - i) + \frac{\gamma}{2} + \delta \right\} \lambda_i \right].
\end{aligned} \tag{30}$$

Here use has been made of eqs. (18) and (21) in deriving the second equality, and of the relation between λ and \vec{r}, \vec{s} obtained from the Young diagram in getting the third equality.

It is clear that applying \hat{H}_n on the singular vectors produces only states at the lower levels, and that the excitation energy is given by the eigenvalue given in (30); $E_{eff} = E_\lambda$. Thus the eigenstates of our system can be written as

$$\hat{\Phi}_\lambda^{GCS} = \hat{J}_\lambda + \sum_{\mu < \lambda} C_\mu \hat{J}_\mu, \tag{31}$$

where \hat{J}_λ is the oscillator representation of the Jack polynomials for the A_{N-1} case (or

the W_N singular vectors) with the coefficients C_μ to be determined from the highest state \hat{J}_λ by the application of \hat{H}_n .

To be more explicit, by taking the inner product of eq. (31) with $\langle J_\nu | \hat{H}_{eff}$, we have

$$\langle J_\nu | \sum_{n>0} \hat{H}_n | J_\lambda \rangle + \sum_{\mu<\lambda} C_\mu \langle J_\nu | \sum_{n>0} \hat{H}_n | J_\mu \rangle = C_\nu (E_\lambda - E_\nu), \quad (\nu < \lambda), \quad (32)$$

which is the master equation to determine the coefficients C_μ successively.⁵ The inner product is easily evaluated by using the oscillator representation given in (29). For example, choosing $\nu = \lambda - 1$ in (32), the second coefficient is found to be

$$C_{\lambda-1} = \frac{\langle J_{\lambda-1} | \hat{H}_1 | J_\lambda \rangle}{E_\lambda - E_{\lambda-1}}, \quad (33)$$

where $\lambda - 1$ stands for the Young diagram with single box removed from λ . Next, setting $\nu = \lambda - 2$, we get an equation for $C_{\lambda-2}$ involving only $C_{\lambda-1}$, which is already known. In this way, all the coefficients can be obtained from (32) successively.

The actual eigenstates in terms of the symmetric power sums (23) can be read off from the explicit expression in terms of the boson oscillators by the rule (25). The total energy is obtained from (6) and (30) as

$$E_0^{GCS} + E_{eff} = \sum_{i=1}^{N_0} \left[\lambda_i + \beta(N_0 - i) + \frac{\gamma}{2} + \delta \right]^2, \quad (34)$$

in agreement with the known results [11].

This completes our procedure to determine the whole eigenfunctions and our elementary derivation of the excitation energy.

As a simple example of the application of our method, we present the results of the eigenstates. For simplicity, we list only those for the D_N case ($\gamma = \delta = 0$) up to level 3:

level 1

$$p_1 = J_{\substack{\vec{r}=(1,0,\dots) \\ \vec{s}=(1,0,\dots)}} ; \quad \square ; \quad 1 + 2\beta(N_0 - 1). \quad (35)$$

level 2

$$\begin{aligned} p_2 - p_1^2 - 4N_0(\beta - 1) &= J_{\substack{\vec{r}=(2,0,\dots) \\ \vec{s}=(1,0,\dots)}} - 4N_0(\beta - 1) ; \quad \square\square ; \quad 2(1 - 3\beta + 2N_0\beta), \\ p_2 + \beta p_1^2 - 8N_0\beta &= J_{\substack{\vec{r}=(1,0,\dots) \\ \vec{s}=(2,0,\dots)}} - 8N_0\beta ; \quad \square\square ; \quad 4(1 - \beta + N_0\beta). \end{aligned} \quad (36)$$

⁵It is easy to check from (30) that the energy difference $E_\lambda - E_\nu > 0$ for $\lambda > \nu$. Hence the energy denominator never vanishes in our eq. (32).

level 3

$$\begin{aligned}
& p_3 - \frac{3}{2}p_1p_2 + \frac{1}{2}p_1^3 + \frac{3(N_0 - 1)(\beta - 1)}{2N_0\beta - 5\beta + 1}p_1 \\
&= J_{\substack{\vec{r}=(3,0,\dots) \\ \vec{s}=(1,0,\dots)}} + \frac{3(N_0 - 1)(\beta - 1)}{2N_0\beta - 5\beta + 1} J_{\substack{\vec{r}=(1,0,\dots) \\ \vec{s}=(1,0,\dots)}} ; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} ; 3(1 - 4\beta + 2N_0\beta), \\
& p_3 + (\beta - 1)p_1p_2 - \beta p_1^3 + \frac{2N_0\beta^2 - 8N_0\beta + 5\beta - 2}{2N_0\beta - 3\beta + 2}p_1 \\
&= J_{\substack{\vec{r}=(2,1,0,\dots) \\ \vec{s}=(1,1,0,\dots)}} + \frac{2N_0\beta^2 - 8N_0\beta + 5\beta - 2}{2N_0\beta - 3\beta + 2} J_{\substack{\vec{r}=(1,0,\dots) \\ \vec{s}=(1,0,\dots)}} ; \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} ; 5 - 8\beta + 6N_0\beta, \quad (37) \\
& p_3 + \frac{3}{2}\beta p_1p_2 + \frac{\beta^2}{2}p_1^3 - \frac{3\beta(N_0\beta + 1)}{N_0\beta - \beta + 2}p_1 \\
&= J_{\substack{\vec{r}=(1,0,\dots) \\ \vec{s}=(3,0,\dots)}} - \frac{3\beta(N_0\beta + 1)}{N_0\beta - \beta + 2} J_{\substack{\vec{r}=(1,0,\dots) \\ \vec{s}=(1,0,\dots)}} ; \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} ; 3(3 - 2\beta + 2N_0\beta).
\end{aligned}$$

Here the corresponding Young diagrams and excitation energies are also exposed.

To summarize, we have given a systematic algorithm to compute eigenfunctions for excited states for the BC_N -CS models. Remarkably these can be easily obtained from those for the A_{N-1} case (but modified to be reflection invariant), which are nothing but singular vectors of the W_N algebra. Our method uses simple oscillator representation, which is easily accesible for physicists.

Acknowledgments

We would like to thank K. Higashijima and H. Suzuki for valuable discussions. Thanks are also due to H. Awata, Y. Matsuo, S. Odake and T. Yamamoto for useful comments. Critical comments by D. Serban on the spinor representations for B_N and D_N cases are also appreciated.

References

- [1] F. Calogero, Jour. Math. Phys. **10** (1969) 2191, 2197; **12** (1971) 419;
 B. Sutherland, Jour. Math. Phys. **12** (1971) 246, 251; Phys. Rev. **A4** (1971) 2019;
A5 (1972) 1372.

- [2] N. Kawakami, Prog. Theor. Phys. **91** (1994) 189;
 F. D. M. Haldane, in the *Correlation Effects in Low Dimensional Electron Systems*,
 eds. A. Okiji and N. Kawakami (Springer, 1994);
 V. Pasquier, preprint SPhT/94-060, hep-th/9405104.
- [3] J. M. Leinaas and J. Myrheim, Phys. Rev. **B37** (1988) 9286;
 A. P. Polychronakos, Nucl. Phys. **B324** (1989) 597;
 Z. N. C. Ha, Phys. Rev. Lett. **73** (1994) 1574; Nucl. Phys. **B435** (1995) 604;
 F. Lesage, V. Pasquier and D. Serban, Nucl. Phys. **B435** (1995) 585.
- [4] N. Kawakami, Phys. Rev. Lett. **71** (1993) 275;
 H. Azuma and S. Iso, Phys. Lett. **B331** (1994) 107;
 M. Stone and M. Fisher, Int. Jour. Mod. Phys. **B8** (1994) 2539.
- [5] K. Hikami and M. Wadati, Phys. Rev. Lett. **73** (1994) 1191.
- [6] R. Stanley, Adv. Math. **77** (1989) 76;
 I. G. Macdonald, Lect. Note in Math. **1271** (Springer, 1987) p. 189.
- [7] A. Jevicki and B. Sakita, Nucl. Phys. **B165** (1980) 511;
 J. A. Minahan and A. P. Polychronakos, Phys. Rev. **B50** (1994) 4236.
- [8] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, Phys. Lett. **B347** (1994) 49; Nucl.
 Phys. **B449** (1995) 347;
 K. Mimachi and Y. Yamada, Comm. Math. Phys. **174** (1995) 447.
- [9] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71** (1981) 313; **94** (1983) 313.
- [10] T. Yamamoto, Jour. Phys. Soc. Japan **63** (1994) 1223.
- [11] D. Bernard, V. Pasquier and D. Serban, Europhys. Lett. **30** (1995) 301.
- [12] V. Fateev and S. Lykhanov, Int. Jour. Mod. Phys. **A3** (1988) 507.